8 per cent greater than the coefficient predicted in the derivation. This error could easily lie within the experimental results since the plot of log N_{Nu_x} vs.

$$\log N_{Pr}{}^2 N_{Gr_x}(1 + 0{\cdot}27\, N_{Gr\frac{1}{2}}/N_{Gr_x})/(0{\cdot}952 + N_{Pr})$$

shows a slight deviation from the predicted power on the group ($N_{Gr_x} + 0{\cdot}27\, N_{Gr\frac{1}{2}}/N_{Gr_x}$). The experimental power found was 0·286 as compared to the predicted power of 0·25.

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EFFECTIVE CONDUCTIVITY FOR CONDUCTION-RADIATION BY TAYLOR SERIES EXPANSION†

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NOMENCLATURE

$E_{b\nu}$, spectral black body emissive power;
E_n, exponential integral function of order n;
k, thermal conductivity;
k_r, radiative conductivity defined by equation (14);
q_r, radiative flux;
q_w, total heat flux;
T, temperature;
α, spectral dependence of the absorption coefficient;
β, state dependence of the absorption coefficient;
ε, emissivity;
κ_R, Rosseland mean absorption coefficient;
θ, dimensionless temperature, T/T_2;
ρ, reflectivity;
ν, frequency;
τ, optical depth, $\int_0^y \beta(y)\,dy$;
τ_0, optical thickness, $\int_0^l \beta(y)\,dy$.

Superscript
*, dimensionless quantity.

Subscript
i, denotes the ith band;
1, denotes the cool boundary;
2, denotes the hot boundary.

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INTRODUCTION

AS A RESULT of intensive study, considerable progress has been achieved over the past decade in understanding energy transfer by combined conduction and radiation in semi-transparent media. Rigorous analyses of radiative transfer characteristically involves either an integral or integrodifferential equation which must be solved numerically as the applicable energy equation, since the radiative flux possesses an integral representation. Several approximations such as the diffusion, differential, substitute kernel, matched asymptotic expansions and others have been used to approximate radiative transfer in gray media. Many of these lose their simplicity and utility when extended to nongray media. Considerable attention has also been given to the "effective" conductivity of radiation with only limited success [1].

The purpose of the present paper is to reexamine the series expansion of the emissive power to develop a workable radiative conductivity for conduction-radiation problams with reflecting boundaries and non-gray participating media. The method was suggested by Viskanta [2] and thoroughly investigated and discussed by Taitel and Hartnett [3] for a gray medium bounded by black walls. This approach results in an approximate solution of a physically rigorous formulation and is advantageous over a numerical or approximate solution of a physically approximate problem. Furthermore, the utility of the method is that it can be applied not only to the planar but to other geometries, boundary conditions and absorption spectra. The present work has been motivated by the need to predict in a simple yet accurate manner heat transfer by combined conduction and radiation in semitransparent solids.

In the analyses [2, 3] the first three terms of the Taylor series expansion are retained in order that the boundary conditions of the differential energy equation may be satisfied. The choice of the number of terms to be retained is arbitrary and can not be determined *a priori* since general solutions for conduction–radiation problems are not available. Comparison with exact solutions is the only means of determining the validity of a given truncation.

ANALYSIS

To illustrate an alternate series expansion approach consider a planar layer of a nongray participating medium. For the absorption characteristics of the medium a number of models such as the exponential wide band [6] and rectangular model [4] have been employed. In this study the spectral absorption coefficient will be represented by the Milne–Eddington model

$$\kappa_\nu(T) = \alpha(\nu)\,\beta(T) \tag{1}$$

with the spectral dependence, $\alpha(\nu) = \alpha_i$, $\nu_{i-1} < \nu \le \nu_i$, which is typical of an amorphous solid [7]. The boundaries of the medium are diffuse, isothermal walls at temperatures T_1 and T_2 having emissivities ε_{1i} and ε_{2i}. The steady state energy balance for combined conduction and radiation in the absence of heat sources can be written in the dimensionless form

$$-\,4N\,\mathrm{d}\theta/\mathrm{d}\tau + q_r^* = q_w^* \tag{2}$$

with the parameter $N = k\beta/4\sigma T_2^3$ physically representing the ratio of the conduction to optically thick radiant transfer. Conduction assures the continuity of temperature at the walls; the boundary conditions are then

$$\theta(0) = \theta_1, \qquad \theta(\tau_0) = \theta_2. \tag{3}$$

The radiative flux is given by

$$q_r^*(\tau) = \theta_1^4 G_1(\tau) - \theta_2^4 G_2(\tau) + \int_0^{\tau_0} H(\tau, t, \theta)\,\theta^4(t)\,\mathrm{d}t \tag{4}$$

where

$$G_1(\tau) = 2\sum_{i=1}^{n} \frac{\varepsilon_{1i} I_i(\theta_1)\,\{E_3(\alpha_i\tau) - 2\rho_{2i}E_3(\alpha_i\tau_0)\,E_3[\alpha_i(\tau_0 - \tau)]\}}{1 - 4\rho_{1i}\rho_{2i}E_3^2(\alpha_i\tau_0)} \tag{5}$$

$$G_2(\tau) = 2\sum_{i=1}^{n} \frac{\varepsilon_{2i} I_i(\theta_2)\,\{E_3[\alpha_i(\tau_0 - \tau)] - 2\rho_{1i}E_3(\alpha_i\tau_0)\,E_3(\alpha_i\tau)\}}{1 - 4\rho_{1i}\rho_{2i}E_3^2(\alpha_i\tau_0)} \tag{6}$$

and

$$H(\tau, t, \theta) = 2\sum_{i=1}^{n} I_i(\theta)\left\{\mathrm{sign}(\tau - t)\,E_2(\alpha_i|\tau - t|) + \frac{\rho_{2i}G_2(\alpha_i\tau)}{\varepsilon_{2i}I_i(\theta_1)}\,E_2(t) - \frac{\rho_{2i}G_2(\alpha_i\tau)}{\varepsilon_{2i}I_i(\theta_2)}\,E_2[\alpha_i(\tau_0 - t)]\right\} \tag{7}$$

in which $I_i(\theta)$ is the fraction of the total emissive power within a band and is readily obtained by integrating Planck's function over the band.

The emissive power may be expanded in a Taylor series about the optical depth which in one dimension has the form

$$\theta^4(t) = \theta^4(\tau) + (t - \tau)\left.\frac{\mathrm{d}\theta^4}{\mathrm{d}t}\right|_{t=\tau} + \frac{(t - \tau)^2}{2}\left.\frac{\mathrm{d}^2\theta^4}{\mathrm{d}t^2}\right|_{t=\tau} + \ldots \tag{8}$$

An approximation to the radiative flux is obtained by substituting the first two terms of equation (8) into equation

(4) and performing the integration with the result

$$q_r^*(\tau) = \theta_1^4 G_1(\tau) - \theta_2^4 G_2(\tau) + \theta^4 H_1(\tau,\theta) + 4\theta^3 H_2(\tau,\theta)\,\mathrm{d}\theta/\mathrm{d}\tau \tag{9}$$

where

$$H_1(\tau,\theta) = 2\sum_{i=1}^{n} I_i(\theta)\Big\{E_3[\alpha_i(\tau_0-\tau)] - E_3(\alpha_i\tau) - \left[\frac{\rho_{1i}G_1(\alpha_i\tau)}{\varepsilon_{1i}I_i(\theta_1)} - \frac{\rho_{2i}G_2(\alpha_i\tau)}{\varepsilon_{2i}I_i(\theta_2)}\right][1-2E_3(\alpha_i\tau_0)]\Big\} \tag{10}$$

$$H_2(\tau,\theta) = 2\sum_{i=1}^{n} J_i(\theta)\Big\{\alpha_i(\tau_0-\tau)E_3[\alpha_i(\tau_0-\tau)] + \alpha_i\tau E_3(\alpha_i\tau) + E_4[\alpha_i(\tau_0-\tau)] + E_4(\alpha_i\tau) - \frac{2}{3} + 2\frac{\rho_{1i}G_1(\alpha_i\tau)}{\varepsilon_{1i}I_i(\theta_1)}\left[\frac{1}{3} - \alpha_i(\tau_0-\tau)E_3(\alpha_i\tau_0) - E_4(\alpha_i\tau_0) - \frac{\alpha_i\tau}{2}\right] + 2\frac{\rho_{2i}G_2(\alpha_i\tau)}{\varepsilon_{2i}I_i(\theta_2)}\left[\frac{1}{3} - \alpha_i\tau E_3(\alpha_i\tau_0) - E_4(\alpha_i\tau_0) - \frac{\alpha_i(\tau_0-\tau)}{2}\right]\Big\} \tag{11}$$

and

$$J_i(\theta) = \frac{1}{\sigma T_2^3}\int_{\nu_{i-1}}^{\nu_i} \frac{\mathrm{d}E_{b\nu}}{\mathrm{d}T}\,\mathrm{d}\nu. \tag{12}$$

This result is not equivalent to the approximation of the radiative flux divergence in the differential energy equation by a three-term expansion [2, 3]. The energy equation then becomes

$$-4N(1-k_r^*)\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = q_w^* - \theta^4 H_1(\tau,\theta) - \theta_1^4 G_1(\tau,\theta) + \theta_2^4 G_2(\tau) \tag{13}$$

where an effective conductivity for radiation

$$k_r^* = 4\theta^3 H_2(\tau,\theta)/N \tag{14}$$

has been introduced.

The two-term Taylor series expansion of θ^4 has resulted in a diffusion formulation with heat sources and sinks for the radiative transfer. This simplification is not possible when additional terms are retained. Numerical solutions can be obtained by any of a number of standard techniques with the total heat flux, q_w^*, being determined such that the boundary conditions are satisfied. The heat flux is then a direct result of the computational procedure.

RADIATIVE CONDUCTIVITY

By employing the Taylor series, a meaningful definition of a radiative conductivity which includes spectral, geometric and wall effects is possible. In the optically thick limit with the absence of walls this conductivity reduces to the classical radiative conductivity, $k_r = 16\sigma T^3/3\kappa_R$. When the medium is transparent in the jth band, $H_{1j} = H_{2j} = 0$, $G_{1j} = \varepsilon_{1j}\varepsilon_{2j}I_j(\theta_1)/1 - \rho_{1j}\rho_{2j}$ and $G_{2j} = \varepsilon_{1j}\varepsilon_{2j}I_j(\theta_2)/1 - \rho_{1j}\rho_{2j}$. The radiative flux predicted by equation (9) is then

$$q_r^* = \frac{I_j(\theta_1)\theta_1^4 - I_j(\theta_2)\theta_2^4}{(1/\varepsilon_{1j}) + (1/\varepsilon_{2j}) - 1}. \tag{15}$$

Following the procedure presented in the analysis "effective" conductivities may be derived for other geometries. Many of the objections to the conductivity concept are therefore eliminated.

DISCUSSION OF RESULTS

To examine the validity of this approach and to compare it to a three-term series, conduction and total heat flux results are presented for black walls in Table 1. It is seen that the total heat flux is predicted accurately with the largest error being approximately 7 per cent for the radiation dominant, self-absorbing case. This is the more critical case since the conductivity correctly reduces to the optically thick and transparent limits as well as the conduction dominant limit. Under this condition, the accuracy is increased by an order of magnitude as compared to the direct application of a three-term expansion.

Although temperature distributions are not presented here, they were found to deviate more from the exact results than the profiles presented by Viskanta [2]. The error in the temperature is readily apparent in the conduction results. When radiation is the predominant mode of energy transfer, either approach is in error since significant higher order terms in the expansion have been neglected. It is of interest to note that the conductivity concept can predict conduction more accurately than the three-term series as is the case for large optical thicknesses.

As the emissivity is decreased the total heat flux is again predicted accurately in the optically thick and thin regimes and in conduction dominant cases as shown in Table 2. No general correlation can be found but the error increases with increasing emissivity when the radiation mode is dominant. When the walls become highly reflective the effect of radiative transfer makes the temperature more uniform in the center of the medium resulting in steep gradients near the walls. This being the case, inaccurate temperature profiles are predicted due to the series truncation as is clearly demonstrated by the conduction heat transfer results.

To examine the conductivity approach as applied to a nongray medium the short wavelength (model A) and long wavelength (model B) window models of Crosbie and Viskanta [4] with black boundaries were selected because exact results are available. Heat transfer results for these models are presented in Table 3. The total heat transfer is predicted

Table 1. Comparison of series expansion approximations for a gray medium bounded by black walls; $\theta_1 = 0{\cdot}5$, $\theta_2 = 1{\cdot}0$

N		$\tau_0 = 0{\cdot}1$		$\tau_0 = 1{\cdot}0$		$\tau_0 = 10{\cdot}0$	
		$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$	$-N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$	$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$
0·01	(1)	0·2460	1·0799	0·1018	0·5675	0·0244	0·1131
	(2)	0·2740 (0·114)†	1·0817 (0·002)†	0·0903 (0·113)	0·6064 (0·069)	0·0156 (0·361)	0·1173 (0·037)
	(3)	0·25 (0·016)†	1·08 (0·0)†	0·106 (0·041)	0·958 (0·688)	0·0129 (0·471)	0·0291 (0·742)
0·1	(1)	2·048	2·880	0·3651	0·7694	0·0749	0·1335
	(2)	2·082 (0·016)	2·883 (0·0)	0·4270 (0·170)	0·7795 (0·013)	0·0739 (0·013)	0·1355 (0·015)
	(3)	2·05 (0·001)	2·87 (0·003)	0·377 (0·033)	0·848 (0·102)	0·074 (0·012)	0·131 (0·018)
1·0	(1)	20·05	20·88	2·203	2·572	0·2856	0·3150
	(2)	20·08 (0·002)	20·88 (0·0)	2·296 (0·042)	2·508 (0·025)	0·2914 (0·020)	0·3156 (0·002)
	(3)	20·1 (0·003)	20·9 (0·001)	1·88 (0·147)	2·23 (0·133)	0·286 (0·001)	0·315 (0·0)
10·0	(1)	200·1	200·9	20·21	20·57	2·093	2·115
	(2)	200·1 (0·0)	200·9 (0·0)	20·31 (0·005)	20·58 (0·0)	2·098 (0·002)	2·115 (0·0)
	(3)	200 (0·0)	201 (0·0)	20·3 (0·005)	20·6 (0·001)	2·09 (0·001)	2·11 (0·002)

(1) Exact numerical results [4].
(2) Predicted by equation (13).
(3) Direct three-term expansion [3].
† Numbers in parentheses are the ratio of the difference of approximate and exact results to the exact results.

Table 2. Effect of the surface emissivity with a gray medium; $\varepsilon_1 = \varepsilon_2$, $\theta_1 = 0{\cdot}5$, $\theta_2 = 1{\cdot}0$

N	ε	$\tau_0 = 0{\cdot}1$		$\tau_0 = 1{\cdot}0$		$\tau_0 = 10{\cdot}0$	
		$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$	$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$	$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$
0·01	0·5	0·2713 (1·331)†	0·5312 (1·014)†	0·1091 (5·452)	0·3894 (1·152)	0·0211(10·567)	0·1090 (1·048)
	0·1	0·2506 (1·251)†	0·2788 (1·044)†	0·1450 (7·239)	0·2178 (1·396)	0·0497(24·875)	0·0911 (1·012)
0·1	0·5	2·078 (1·039)	2·332 (1·000)	0·4282 (2·144)	0·5698 (1·247)	0·0909 (4·543)	0·1288 (1·056)
	0·1	2·052 (1·026)	2·079 (1·000)	0·3958 (1·979)	0·4172 (1·062)	0·1096 (5·477)	0·1187 (1·032)
1·0	0·5	20·08 (1·004)	20·33 (1·000)	2·270 (1·136)	2·370 (0·989)	0·2980 (1·491)	0·3104 (1·011)
	0·1	20·05 (1·003)	20·08 (1·000)	2·212 (1·106)	2·225 (0·991)	0·3027 (1·514)	0·3053 (1·028)
10·0	0·5	200·1 (1·001)	200·3 (1000.)	20·28 (1·014)	20·37 (0·999)	2·101 (1·050)	2·110 (1·000)
	0·1	200·1 (1·001)	200·1 (1·001)	20·21 (1·000)	20·22 (0·999)	2·104 (1·052)	2·106 (1·000)

† Numbers in parentheses are the ratio of approximate to exact results [5].

Table 3. Series expansion results for a nongray medium; $\varepsilon = 1{\cdot}0$, $\theta_1 = 0{\cdot}5$, $\theta_2 = 1{\cdot}0$

N	Model	$\tau_0 = 0{\cdot}1$		$\tau_0 = 1{\cdot}0$		$\tau_0 = 10{\cdot}0$	
		$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$	$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$	$-4N\, d\theta/d\tau\|_0$	$q_w/\sigma T_2^4$
0·01	A	0·2202 (0·033)†	1·1170 (0·0)†	0·0446 (0·174)	0·8301 (0·012)	0·0061 (0·456)	0·6420 (0·002)
	B	0·2580 (0·102)†	1·1021 (0·001)†	0·1372 (0·513)	0·7321 (0·027)	0·0236 (0·018)	0·4148 (0·005)
0·1	A	2·021 (0·004)	2·917 (0·0)	0·2672 (0·056)	1·0075 (0·001)	0·0383 (0·036)	0·6600 (0·001)
	B	2·061 (0·008)	2·903 (0·0)	0·4002 (0·216)	0·9070 (0·003)	0·0769 (0·130)	0·4331 (0·002)
1·0	A	20·02 (0·003)	20·92 (0·0)	2·081 (0·011)	2·807 (0·0)	0·2279 (0·006)	0·8401 (0·0)
	B	20·07 (0·002)	20·90 (0·0)	2·221 (0·034)	2·711 (0·003)	0·2679 (0·017)	0·6130 (0·001)
10·0	A	200·0 (0·0)	200·9 (0·0)	20·08 (0·001)	20·81 (0·0)	2·029 (0·001)	2·640 (0·0)
	B	200·1 (0·0)	200·9 (0·0)	20·22 (0·003)	20·71 (0·003)	2·069 (0·002)	2·413 (0·0)

† Numbers in parentheses are the ratio of the difference of approximate and exact results to exact results [4].

to within less than 3 per cent error over the entire range covered. Since the absorption spectrum contains transparent regions in which radiation does not influence the temperature profile, the improved accuracy is to be expected. This finding is significant since it demonstrates that results of engineering accuracy for realistic materials can be predicted.

CONCLUSIONS

The two-term Taylor series expansion of the emissive power has been shown to be an accurate and useful method for the prediction of the net heat transfer for combined radiation and conduction. The approximation leads to a meaningful definition of the radiative conductivity which is extremely useful when radiation is coupled with other modes of heat transfer since the radiation can then be treated as a diffusion process with heat sources and sinks. This "effective" conductivity has a wider range of applicability than the classical radiative conductivity since it accounts for spectral as well as wall effects and is exact in the transparent and optically thick limits. It has been shown to accurately predict the total heat flux under a variety of boundary conditions and for nongray as well as gray media.

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THIN LIQUID FILMS UNDER SIMULTANEOUS SHEAR AND GRAVITY FORCES

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NOMENCLATURE

a, acceleration due to body forces;
Re_L, film Reynolds number $4\Gamma/\mu_L$;
u, local liquid film velocity;
u^+, u/u_*;
u_*, shear velocity $(\tau_w/\rho_L)^{\frac{1}{2}}$;
u_{*i}, interfacial shear velocity $(\tau_i/\rho_L)^{\frac{1}{2}}$;
y, distance from wall;
y^+, yu_*/ν_L;
β, shear parameter $(u_{*i}^3/a\nu_L)^{\frac{2}{3}}$;
Γ, mass flow rate per unit wetted perimeter;
δ, mean film thickness;
δ^+, non-dimensional film thickness $\delta u_*/\nu_L$;
η, non-dimensional film thickness $\delta(a/\nu_L^2)^{\frac{1}{3}}$;
μ_L, dynamic viscosity of liquid;
ν_L, kinematic viscosity of liquid;
ρ_L, ρ_v, density of liquid and vapor respectively;
τ_w, wall shear;
τ_i, interfacial shear.

INTRODUCTION

A COMMON feature of many two phase processes involves the transport of heat or of mass across a flowing thin liquid film. One may cite climbing film evaporators, condensers, boiling-water nuclear reactors and some desalination